Instability of a surface of discontinuity of velocity in a parallel uniform magnetic field

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The problem described by the title is investigated when the magnetic field is uniform and parallel to the velocity on the two sides of a surface of discontinuity of velocity in an electrically conducting inviscid fluid. The secular equation depends on two parameters β and N, where β is the ratio of magnetic Reynolds number to dimensionless wave number and N is the ratio of the magnetic to the kinetic energy of the fluid. It is found that the flow is unstable for all values of β and N.

1. Introduction

The instability of the common surface in a uniform flow of two fluids known as Helmholtz flow is well known. According to recent investigations a magnetic field has been found to exercise a strong stabilizing influence in many unstable flows. Drazin (1960*a*) has found that a jet of single fluid with a parallel magnetic field is unstable at zero magnetic Reynolds number, however large the magnetic field may be. In a later paper (1960*b*) he has confirmed that the flow in the two-fluid model is unstable to long-wave disturbances for all finite magnetic Reynolds number.

Michael (1955) found the Helmholtz flow of perfectly conducting inviscid fluid in a parallel magnetic field to be stable or unstable according as $N >$ or < 1, where N is the ratio of the magnetic to the kinetic energy of the fluid. Drazin refers to an unpublished report and to an unpublished paper of I. C. T. Nisbet on the stability of Helmholtz flow of inviscid fluids with finite conductivity in a parallel magnetic field which as far as the author knows has not yet appeared in print. The author has for some time been engaged in the study of the stability of Helmholtz flow under different conditions and the present paper sets forth some of his results on the stability of Helmholtz flow in two incompressible inviscid electrically conducting fluids with a uniform magnetic field parallel to the flow, the velocities of the flow being equal and opposite. The boundary conditions considered in this paper are somewhat more extended than those taken into account by Michael, but are similar to those of Drazin (1960*b*). The stability conditions are quite complex. A simple differentiation of these conditions is possible through a parameter $\beta = R_M/\alpha_1$, where R_M is the magnetic Reynolds number and α_1 a quantity which may be called a dimensionless wave number. For instance we can distinguish between three cases:

(i) $\beta \rightarrow \infty$, when the flow is found to be unstable for all N. This contradicts

Michael's result when $R_M = \infty$. The reason for this discrepancy is probably due to the fact that the boundary conditions used in this paper and the order of the differential equation are different from those of Michael.

(ii) $\beta = 0$, when it has been found that the motion is also unstable.

(iii) $0 < \beta < \infty$, when the motion is also found to be unstable for all N.

Further it has been found that the flow is unstable however large the magnetic field may be. An exception to the rule of stabilizing influence of magnetic fields is therefore confirmed in this paper.

2. Formulation of the problem and the equation of hydromagnetic stability

We take axes $Oxyz$ such that Ox is parallel to the velocity and field vectors at the interface and Oy is perpendicular to the interface, Oz being perpendicular to Ox and Oy on the interface. With reference to these axes the velocity and magnetic fields are given by

$$\mathbf{q} = \begin{cases} (U, 0, 0) & \text{for} \quad y < 0, \\ (-U, 0, 0) & \text{for} \quad y > 0, \end{cases}$$

$$\mathbf{H} = (H_0, 0, 0) \quad \text{for} \quad y < 0 \quad \text{and} \quad y > 0.$$

We shall consider the stability of the equilibrium of the interface to small two-dimensional disturbances in the (x, y)-plane. Following the usual methods of hydrodynamic stability, we shall substitute

$$u_i = \left.\begin{cases} (U + u_0, v_0, 0) & \text{for} \quad y < 0 \\ (-U + u_1, v_1, 0) & \text{for} \quad y > 0 \end{cases}\right\} \quad (i = 1, 2, 3),$$

and

$$H_i = \begin{cases} (H_0 + h_0, k_0, 0) & \text{for} \quad y < 0 \\ (H_0 + h_1, k_1, 0) & \text{for} \quad y > 0 \end{cases}$$

into the hydromagnetic equations for a homogeneous incompressible inviscid fluid of density ρ, magnetic permeability μ, electrical conductivity σ and magnetic diffusivity $\lambda = 1/4\pi\mu\sigma$, which are given by

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} - \frac{\mu}{4\pi\rho} H_j \frac{\partial H_i}{\partial x_j} = -\frac{\partial \varpi}{\partial x_i} \quad (i = 1, 2, 3), \tag{2.1}$$

$$\frac{\partial H_i}{\partial t} + u_j \frac{\partial H_i}{\partial x_j} - H_j \frac{\partial u_i}{\partial x_j} = \lambda \nabla^2 H_i \quad (i = 1, 2, 3), \tag{2.2}$$

$$\partial u_i / \partial x_i = 0, \tag{2.3}$$

$$\partial H_i / \partial x_i = 0, \tag{2.4}$$

where

$$\varpi = \frac{1}{\rho}\left(p + \frac{\mu H^2}{8\pi}\right)$$

and p is the hydrodynamic pressure. The equilibrium of the interface in the undisturbed state is maintained by the continuity of total pressure (i.e. stress) across the surface, i.e. if ϖ_0 is the value of ϖ in the undisturbed state, then ϖ_0 is continuous across the interface. In a small disturbance of the above type let the disturbed ϖ be $\varpi_0 + \omega_0$ for $y < 0$, and $\varpi_0 + \omega_1$ for $y > 0$. Then we shall linearize equations (2.1)–(2.4) by neglecting the squares and products of small quantities and write equations for $y < 0$ as follows:

$$\frac{\partial u_0}{\partial t} + U\frac{\partial u_0}{\partial x} - \frac{\mu H_0}{4\pi\rho}\frac{\partial h_0}{\partial x} = -\frac{\partial \omega_0}{\partial x}, \tag{2.5}$$

$$\frac{\partial v_0}{\partial t} + U\frac{\partial v_0}{\partial x} - \frac{\mu H_0}{4\pi\rho}\frac{\partial k_0}{\partial x} = -\frac{\partial \omega_0}{\partial y}, \tag{2.6}$$

$$\frac{\partial h_0}{\partial t} + U\frac{\partial h_0}{\partial x} - H_0\frac{\partial u_0}{\partial x} = \lambda\nabla^2 h_0, \tag{2.7}$$

$$\frac{\partial k_0}{\partial t} + U\frac{\partial k_0}{\partial x} - H_0\frac{\partial v_0}{\partial x} = \lambda\nabla^2 k_0, \tag{2.8}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0, \tag{2.9}$$

$$\frac{\partial h_0}{\partial x} + \frac{\partial k_0}{\partial y} = 0. \tag{2.10}$$

The continuity equation (2.9) and equation (2.10) suggest that there exist functions $\psi(x, y, t)$, $\chi(x, y, t)$ such that

$$u_0 = -\partial\psi/\partial y, \quad v_0 = \partial\psi/\partial x, \tag{2.11}$$

$$h_0 = -\partial\chi/\partial y, \quad k_0 = \partial\chi/\partial x. \tag{2.12}$$

We shall now consider a disturbance of the type

$$\psi = \phi(y)\, e^{i\alpha(x-ct)}, \quad \chi = \theta(y)\, e^{i\alpha(x-ct)}, \quad \omega_0 = S(y)\, e^{i\alpha(x-ct)}, \tag{2.13}$$

where $c = c_r + ic_i$ is a complex wave velocity and α a positive wave number. If ultimately c_i is found to be negative or zero the motion is stable, and for $c_i > 0$ the motion in unstable. With the help of (2.13), (2.11) and (2.12) equations (2.5)–(2.8) become

$$(U-c)\,\phi' - (\mu H_0/4\pi\rho)\,\theta' = S, \tag{2.14}$$

$$(U-c)\,\phi - (\mu H_0/4\pi\rho)\,\theta = S'/\alpha^2, \tag{2.15}$$

$$i\alpha[(U-c)\,\theta' - H_0\phi'] = \lambda(\theta''' - \alpha^2\theta'), \tag{2.16}$$

$$i\alpha[(U-c)\,\theta - H_0\phi] = \lambda(\theta'' - \alpha^2\theta), \tag{2.17}$$

where accents denote differentiation with respect to y. Equation (2.16) is not independent as it can be deduced from (2.17) only by a single differentiation and so we drop it. Eliminating S between (2.14) and (2.15) we have

$$(U-c)\,(D^2-\alpha^2)\,\phi = (\mu H_0/4\pi\rho)\,(D^2-\alpha^2)\,\theta. \tag{2.18}$$

From (2.17)

$$i\alpha(U-c)\,\theta - \lambda(D^2-\alpha^2)\,\theta = i\alpha H_0\phi, \tag{2.19}$$

where

$$D \equiv d/dy. \tag{2.20}$$

Eliminating ϕ between (2.18) and (2.19) we have

$$\begin{aligned} 0 &= \left[\lambda(D^2-\alpha^2) + \frac{i\alpha\mu H_0^2}{4\pi\rho(U-c)} - i\alpha(U-c)\right](D^2-\alpha^2)\,\theta \\ &= (D^2-\alpha^2)\left[\lambda(D^2-\alpha^2)\,\theta + i\alpha\left\{\frac{\mu H_0^2 - 4\pi\rho(U-c)^2}{4\pi\rho(U-c)}\right\}\theta\right]. \end{aligned} \tag{2.21}$$

3. Boundary conditions

The conditions which must be satisfied on the common surface are

(i) the total pressure, $p+\mu H^2/8\pi$, must be continuous, i.e. S is continuous at the interface;

(ii) the normal velocity on each side must be equal to the normal velocity of the interface;

(iii) the Maxwell equation $\nabla.\mathbf{H} = 0$ implies that the normal component of the magnetic field must be continuous;

(iv) the tangential component of the magnetic field must be continuous as the interface is not a current sheet.

Let $y = \eta(x,t)$ be the displacement of the interface at time t after the disturbance. We shall assume η to be small such that we can neglect $(\partial\eta/\partial x)^2$. The condition (i) implies that across the interface $y = \eta(x,t)$

$$[(U-c)\,\phi' - (\mu H_0/4\pi\rho)\,\theta'] \quad \text{is continuous.} \tag{3.1}$$

The condition (ii) implies that on $y = \eta(x,t)$

$$v_0 - U\partial\eta/\partial x = v_1 + U\partial\eta/\partial x = \partial\eta/\partial t,$$

i.e.
$$v_0 - Ui\alpha\eta = v_1 + Ui\alpha\eta = -i\alpha c\eta,$$

since $\eta(x,t) \propto e^{i\alpha(x-ct)}$. Hence we have

$$(U+c)\,v_0 = -(U-c)\,v_1 \quad \text{on} \quad y = \eta(x,t). \tag{3.2}$$

The condition (iii) implies that on $y = \eta(x,t)$

$$k_0 - H_0(\partial\eta/\partial x) = k_1 - H_0(\partial\eta/\partial x),$$

or
$$k_0 = k_1.$$

Hence we can say that

$$\theta(y) \quad \text{is continuous on} \quad y = \eta(x,t). \tag{3.3}$$

The condition (iv) implies that, on

$$y = \eta(x,t), \quad h_0 = h_1, \tag{3.4}$$

or we can say that θ' is continuous on

$$y = \eta(x,t).$$

The disturbance must be bounded at infinity. This implies

$$(\theta,\phi) \to 0 \quad \text{as} \quad \begin{cases} y \to +\infty & \text{for} \quad y > 0 \quad \text{solution,} \\ y \to -\infty & \text{for} \quad y < 0 \quad \text{solution.} \end{cases} \tag{3.5}$$

4. Solution of the problem

The solution of the differential equation (2.21) is given by

$$\theta = \theta_1 + \theta_2, \tag{4.1}$$

where θ_1 and θ_2 satisfy the following differential equations

$$(D^2-\alpha^2)\,\theta_1 = 0, \quad \lambda(D^2-\alpha^2)\,\theta_2 + i\alpha\left\{\frac{\mu H_0^2 - 4\pi\rho(U-c)^2}{4\pi\rho(U-c)}\right\}\theta_2 = 0, \tag{4.2}$$

of which the solutions are

$$\theta_1 = A_0 e^{\alpha y} + B_0 e^{-\alpha y}; \quad \theta_2 = A_1 e^{(m-in)y} + B_1 e^{-(m-in)y} \quad (m > 0), \tag{4.3}$$

where
$$(m-in)^2 = \alpha^2 - i\alpha \frac{\mu H_0^2 - 4\pi\rho(U-c)^2}{4\pi\rho\lambda(U-c)} \quad (m > 0), \tag{4.4}$$

$$\lambda = 1/4\pi\mu\sigma. \tag{4.5}$$

The boundary condition (3.5) for the $y < 0$ solution suggests that $B_0 = B_1 = 0$. Hence the solution for θ becomes

$$\theta = \theta_1 + \theta_2 = A_0 e^{\alpha y} + A_1 e^{(m-in)y} \quad (m > 0). \tag{4.6}$$

From (2.19) after making use of (4.4) and (4.6) we have

$$\phi = \frac{(U-c)}{H_0} A_0 e^{\alpha y} + \frac{\mu H_0}{4\pi\rho(U-c)} A_1 e^{(m-in)y} \quad (m > 0). \tag{4.7}$$

The corresponding values of θ and ϕ for $y > 0$ which can be obtained from those of θ and ϕ for $y < 0$ (i.e. from (4.4) to (4.7)) simply by replacing U by $-U$ are given by

$$\theta = A_0' e^{-\alpha y} + A_1' e^{-(m'-in')y} \quad (m' > 0), \tag{4.8}$$

$$\phi = -\frac{(U+c)}{H_0} A_0' e^{-\alpha y} - \frac{\mu H_0}{4\pi\rho(U+c)} A_1' e^{-(m'-in')y} \quad (m' > 0), \tag{4.9}$$

where
$$(m'-in')^2 = \alpha^2 + i\alpha \frac{\mu H_0^2 - 4\pi\rho(U+c)^2}{4\pi\rho\lambda(U+c)} \quad (m' > 0). \tag{4.10}$$

Since $|A_0|$, $|A_1|$, $|A_0'|$, $|A_1'|$ are small, we can replace $e^{\pm\alpha\eta}$, $e^{(m-in)\eta}$, $e^{-(m'-in')\eta}$, by 1. This means that when we determine the unknown constants A_0, A_0', A_1, A_1' by using the boundary conditions at the interface $y = \eta$ we take the boundary conditions (3.1)–(3.4) at $y = 0$ (i.e. at the undisturbed position of the interface) instead of at $y = \eta(x,t)$. Therefore the boundary conditions (3.1)–(3.4) at $y = 0$ give the following relations between the unknown constants A_0, A_1, A_0', A_1':

$$-\left[\frac{\mu H_0}{4\pi\rho} - \frac{(U-c)^2}{H_0}\right] A_0 = \left[\frac{\mu H_0}{4\pi\rho} - \frac{(U+c)^2}{H_0}\right] A_0', \tag{4.11}$$

$$(U+c)\left[\frac{(U-c)}{H_0} A_0 + \frac{\mu H_0}{4\pi\rho(U-c)} A_1\right] = (U-c)\left[\frac{(U+c)}{H_0} A_0' + \frac{\mu H_0}{4\pi\rho(U+c)} A_1'\right], \tag{4.12}$$

$$A_0 + A_1 = A_0' + A_1', \tag{4.13}$$

$$A_0\alpha + A_1(m-in) = -A_0'\alpha - A_1'(m'-in'). \tag{4.14}$$

For a non-trivial solution to exist we have on eliminating the constants A_0, A_1, A_0', A_1' from (4.11) to (4.14)

$$\frac{\mu H_0^2}{4\pi\rho}\alpha 8U^2c^2 = \left(\frac{\mu H_0^2}{4\pi\rho} - U^2 - c^2\right) \times \left[(U-c)^2(m-in)\left\{\frac{\mu H_0^2}{4\pi\rho} - (U+c)^2\right\} + (U+c)^2(m'-in')\left\{\frac{\mu H_0^2}{4\pi\rho} - (U-c)^2\right\}\right]. \tag{4.15}$$

We shall now introduce dimensionless quantities α_1, c_1, R_M, N and β defined by

$$\alpha_1 = \alpha l,\ c_1 = c/U,\ R_M = Ul/\lambda,\ N = \mu H_0^2/(4\pi\rho U^2),\ \beta = R_M/\alpha_1, \tag{4.16}$$

where l is a characteristic length. In terms of these dimensionless parameters the secular equation (4.15) becomes

$$\begin{aligned} 8Nc_1^2 = (N-1-c_1^2)\,[(1+c_1)^2\{N-(1-c_1)^2\}(a'-ib') \\ +(1-c_1)^2\{N-(1+c_1)^2\}(a-ib)], \end{aligned} \tag{4.17}$$

or equivalently

$$0 = \{N-(1-c_1)^2\}\{N-(1+c_1)^2\}\left[\frac{2(1+c_1^2)}{N-1-c_1^2}+\frac{i\beta(1+c_1)}{a'-ib'+1}-\frac{i\beta(1-c_1)}{a-ib+1}\right], \tag{4.17a}$$

where

$$\left.\begin{aligned} a-ib = (m-in)/\alpha = \pm[1-\{i\beta/(1-c_1)\}\{N-(1-c_1)^2\}]^{\frac{1}{2}} \quad (a>0), \\ a'-ib' = (m'-in')/\alpha = \pm[1+\{i\beta/(1+c_1)\}\{N-(1+c_1)^2\}]^{\frac{1}{2}} \quad (a'>0). \end{aligned}\right\} \tag{4.18}$$

The signs before the square roots in the expression for $a-ib$ and $a'-ib'$ will be taken in such a way that a and a' become positive. Before considering the general case we shall first of all study three particular cases namely (i) when $\beta \to \infty$, (ii) when $N \to \infty$ and (iii) when $\beta = 0$.

Case (i) when $\beta \to \infty$

(*a*) *For finite N.* In this case (4.18) and (4.17) respectively take the following forms:

$$\left.\begin{aligned} a-ib &= \pm\left[\frac{i\beta}{1-c_1}\{(1-c_1)^2-N\}\right]^{\frac{1}{2}} = \pm\left[\frac{i}{1-c_1}\{(1-c_1)^2-N\}\right]^{\frac{1}{2}}\beta^{\frac{1}{2}} = (d-ie)\,\beta^{\frac{1}{2}}, \\ a'-ib' &= \pm\left[\frac{i\beta}{1+c_1}\{N-(1+c_1)^2\}\right]^{\frac{1}{2}} = \pm\left[\frac{i}{1+c_1}\{N-(1+c_1)^2\}\right]^{\frac{1}{2}}\beta^{\frac{1}{2}} = (d'-ie')\,\beta^{\frac{1}{2}} \end{aligned}\right\} \tag{4.19}$$

with $d > 0$, $d' > 0$; and

$$\begin{aligned} 0 &= (N-1-c_1^2)\,[(1+c_1)^2\{N-(1-c_1)^2\}(d'-ie')+(1-c_1)^2\{N-(1+c_1)^2\}(d-ie)] \\ &= (N-1-c_1^2)\{N-(1+c_1)^2\}\{N-(1-c_1)^2\}\left[\frac{i(1+c_1)}{d'-ie'}-\frac{i(1-c_1)}{d-ie}\right], \end{aligned} \tag{4.20}$$

which can be obtained directly from (4.17*a*). The factor

$$(N-1-c_1^2)\{N-(1+c_1)^2\}\{N-(1-c_1)^2\}$$

when equated to zero gives six roots for c_1, i.e.

$$c_1 = \pm(N^{\frac{1}{2}}\pm 1), \quad \pm(N-1)^{\frac{1}{2}}, \tag{4.21}$$

which can be arranged as follows

$$\left.\begin{aligned} c_1 = \pm(N^{\frac{1}{2}}\pm 1), \quad \pm(N-1)^{\frac{1}{2}} \quad \text{for} \quad N \geqslant 1, \\ c_1 = \pm(N^{\frac{1}{2}}\pm 1), \quad \pm i(1-N)^{\frac{1}{2}} \quad \text{for} \quad N \leqslant 1. \end{aligned}\right\} \tag{4.21a}$$

To find out the roots of

$$\{i(1+c_1)/(d'-ie')\}-\{i(1-c_1)/(d-ie)\} = 0$$

for c_1 we write it in the form

$$(1-c_1)/(d-ie) = (1+c_1)/(d'-ie') \tag{4.21b}$$

and square both sides to give

$$(1-c_1)^3[N-(1+c_1)^2] = (1+c_1)^3[(1-c_1)^2-N]. \tag{4.22}$$

The roots of this equation are

$$c_1 = \pm[1+\tfrac{3}{2}N \pm (\tfrac{9}{4}N^2+4N)^{\frac{1}{2}}]^{\frac{1}{2}},$$

which can be expressed in the following way

$$\left.\begin{aligned} c_1 &= \pm[1+\tfrac{3}{2}N+(\tfrac{9}{4}N^2+4N)^{\frac{1}{2}}]^{\frac{1}{2}} && \text{for} \quad N <, =, > 1; \\ c_1 &= \pm[1+\tfrac{3}{2}N-(\tfrac{9}{4}N^2+4N)^{\frac{1}{2}}]^{\frac{1}{2}} && \text{for} \quad N \leqslant 1 \\ &= \pm i[(\tfrac{9}{4}N^2+4N)^{\frac{1}{2}}-1-\tfrac{3}{2}N]^{\frac{1}{2}} && \text{for} \quad N \geqslant 1. \end{aligned}\right\} \tag{4.22a}$$

Of the four roots given by (4.22*a*) the roots which satisfy equation (4.21*b*) in such a way that d and d' become positive are given by (verified by giving numerical values to N, namely $N = 0{\cdot}1, 0{\cdot}2, 0{\cdot}5, 1, 2, 4, 6$)

$$\left.\begin{aligned} c_1 &= \pm[1+\tfrac{3}{2}N-(\tfrac{9}{4}N^2+4N)^{\frac{1}{2}}]^{\frac{1}{2}} && \text{for} \quad N \leqslant 1, \\ c_1 &= +i[(\tfrac{9}{4}N^2+4N)^{\frac{1}{2}}-1-\tfrac{3}{2}N]^{\frac{1}{2}} && \text{for} \quad N \geqslant 1. \end{aligned}\right\} \tag{4.23}$$

(*b*) *When N is not finite but comparable with β as $\beta \to \infty$.* In this case equation (4.17) is modified as follows

$$0 = (1-c_1)^2(f-ig)+(1+c_1)^2(f'-ig'), \tag{4.24}$$

where $$f-ig = \pm[-i\gamma/(1-c_1)]^{\frac{1}{2}}, \quad f'-ig' = \pm[i\gamma/(1+c_1)]^{\frac{1}{2}}, \tag{4.24a}$$

with $f > 0, f' > 0$, and

$$\gamma = \beta/N = 4\pi\mu U\sigma/\alpha N. \tag{4.25}$$

Removing the square roots which occur in $f-ig$ and $f'-ig'$ of (4.24) by squaring we obtain the equation $1+3c_1^2 = 0$ of which the roots for c_1 are

$$c_1 = \pm i/\sqrt{3}. \tag{4.26}$$

Of the two roots given by (4.26), satisfying conditions $f > 0, f' > 0$, only one root satisfies equation (4.24) and is given by

$$c_1 = +i/\sqrt{3}. \tag{4.27}$$

Case (ii) when $N \to \infty$

In this case equation (4.17) reduces to (4.24) where γ occurring in $f-ig$ and $f'-ig'$ is to be replaced by β and as before $c_1 = +i/\sqrt{3}$ will satisfy the modified secular equation with $f > 0, f' > 0$ (after replacing γ by β).

Case (iii) when $\beta = 0$

In this case $a-ib = 1 = a'-ib'$ and equation (4.17) becomes

$$0 = (1+c_1^2)\{N-(1-c_1)^2\}\{N-(1+c_1)^2\}, \tag{4.28}$$

which gives $$c_1 = \pm(N^{\frac{1}{2}} \pm 1), \quad \pm i. \tag{4.29}$$

This corresponds to the classical solution when there is no magnetic field (i.e. $R_M = N = 0$).

General case

The roots of the secular equation (4.17*a*) will be determined by the following equations
$$\{N-(1-c_1)^2\}\{N-(1+c_1)^2\} = 0 \tag{4.30}$$
and
$$\frac{2(1+c_1^2)}{N-1-c_1^2}+\frac{i\beta(1+c_1)}{a'-ib'+1}-\frac{i\beta(1-c_1)}{a-ib+1} = 0. \tag{4.31}$$

The roots of equation (4.30) are given by $c_1 = \pm(N^{\frac{1}{2}}\pm 1)$. We shall study equation (4.31) and see how far it is possible to discuss the behaviour of its roots without laborious numerical calculation. Removing the square roots which occur in $a-ib$ and $a'-ib'$ given by (4.18) by repeated squaring and putting $c_1 = ix$, we get an algebraic equation of 16th degree for x with real coefficients containing N and β as parameters,

$$\begin{aligned}&\beta^2(N-1+x^2)^4[x^4+(3N+2)x^2-(N-1)]^2\\&\quad+8\beta Nx(N-1+x^2)^2[-N^2(3x^2-1)(1-x^2)+(1-x^4)^2+2N(1+x^2)^2(2x^2-1)]\\&\quad-[64N^4x^4-16N^2x^2(1+x^2)^2(N-1+x^2)^2] = 0\end{aligned} \tag{4.32}$$

(when $\beta\to\infty$ for finite and infinite N as we have considered in case (i), equation (4.32) reduces to (4.22), $x^2-(1-N) = 0$ and the equation $3x^2-1 = 0$, respectively). Equation (4.32) is a quadratic in β whose roots are given by

$$\begin{aligned}\beta &= \frac{4Nx}{(N-1+x^2)^2[x^4+(3N+2)x^2-(N-1)]^2}[-\{x^8+4Nx^6+(3N^2+6N-2)x^4\\&\qquad-4N^2x^2+(N-1)^2\}\pm 2x(1+x^2)\{(1+x^2)(-x^6-(2N-1)x^4\\&\qquad+(3N^2+1)x^2-(N-1)^2)\}^{\frac{1}{2}}]\\&=f(x,N),\quad\text{say.}\end{aligned} \tag{4.33}$$

(The eigen values $c_1 = \pm(N^{\frac{1}{2}}\pm 1)$ correspond to null eigen vectors. This was pointed out by Prof. S. Chandrasekhar.)

(i) $N>1$. In this case the curve $\beta = f(x,N)$ in the (x,β)-plane has a cusp at
$$[x_0 = \{(\tfrac{9}{4}N^2+4N)^{\frac{1}{2}}-1-\tfrac{3}{2}N\}^{\frac{1}{2}},\infty].$$

The curve $\beta = f(x,2)$ has been drawn in figure 1. For other values of N the corresponding curves are similar in nature to that for $N = 2$. Hence from the nature of the curve $\beta = f(x,2)$ we can say that equation (4.32) for x has two real roots beyond a particular value of β, say β_0, and has four real roots for $0\leqslant\beta\leqslant\beta_0$. Further it can be proved that equation (4.32) has no purely imaginary root. Hence the remaining roots of equation (4.32) are complex. Of the real roots only one root, which lies in the closed interval $x_0\leqslant x\leqslant 1$ with the corresponding β, satisfies equation (4.31) with $a>0$, $a'>0$. This result has been verified for $N = 2$.

(ii) $N = 1$. In this case the curve $\beta = f(x,1)$ in the (x,β)-plane having a cusp at $(0,\infty)$ has been drawn in figure 1. From the figure we see that in this case the equation for x has two real roots (one positive and one negative) for any particular value of β and it has been found that both the roots, which lie in the closed interval $-1{\cdot}2496 \leqslant x \leqslant 1$, with the corresponding β satisfy equation (4.31) with $a > 0$, $a' > 0$. The remaining roots are all complex.

(iii) $N < 1$. In this case the curve $\beta = f(x, N)$ in the (x, β)-plane has two cusps at $(\pm(1-N)^{\frac{1}{2}}, \infty)$. The curve $\beta = f(x, \frac{1}{2})$ has been drawn in figure 1. For other values of N the corresponding curves are similar in nature to that of $N = \frac{1}{2}$. From the nature of the curve $\beta = f(x, \frac{1}{2})$ we can say that equation (4.32) for x

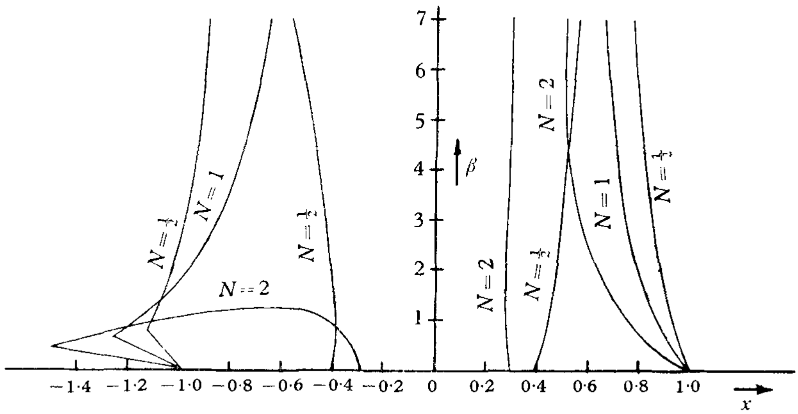

FIGURE 1

has four real roots (two positive and two negative) for a particular finite value of β. The remaining roots are all complex. For a particular value of β only two real roots (one positive and one negative) which are obtained from the part of the curve $\beta = f(x, N)$ passing through the points $(\pm 1, 0)$ and $(\pm(1-N)^{\frac{1}{2}}, \infty)$ satisfy equation (4.31) with the conditions $a > 0$, $a' > 0$.

5. Conclusion

In this section we summarize the results obtained by us in §4. As stated before, the dimensionless complex wave velocity $c_1 = c_{1r} + ic_{1i}$ determined by the secular equation (4.17) or (4.17*a*) containing β and N as parameters will give us information regarding stability. If c_{1i} is zero or negative the motion will be stable, otherwise (i.e. if c_{1i} is positive) the motion will be unstable. We have the following cases:

(*a*) $\beta \to \infty$, then (i) for all finite N (the ratio of magnetic to kinetic energy of the fluid), (4.21*a*) and (4.23) show that the motion in unstable, (ii) when N becomes infinite of the order of β, (4.27) shows that the motion is also unstable. β may tend to infinity either by $R_M \to \infty$ keeping α_1 fixed, or by $\alpha_1 \to 0$, R_M remaining fixed. The former case ($\sigma = \infty$) was considered by Michael whose result has been stated in the Introduction. In this case we conclude that our results contradict Michael's results for $N > 1$. The latter case (i.e. R_M remaining fixed, $\alpha_1 \to 0$) corresponds to long-wave disturbances.

(*b*) $\beta = 0$, i.e. when $R_M = 0$ ($\sigma = 0$), then (4.29) shows that the motion is always unstable.

(*c*) $0 < \beta < \infty$, the motion is always unstable for all N.

(*d*) $N \to \infty$, then from the result obtained in case (ii) we conclude that the motion is unstable for all β. Hence we can say that the Helmholtz flow is also unstable however large the magnetic field may be.

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